DIFFERENTIAL MANIFOLDS HW 4

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1. Exercise 1.36

Suppose that (a, U_a) is an atlas on M. Then (using the book's convention), given $p \in M$ we can find $U_a \ni p$. Consider the image $a(p) \in \mathbb{R}^n$. Since $a(U_a)$ is open, we can find $\epsilon_p > 0$ such that $B(a(p), \epsilon_p)$ (ball of radius ϵ_p centered at $a(p)$) such that $B(a(p), \epsilon_p) \subset a(U_a)$.

Consider $\phi_p : a^{-1}(B(a(p), \epsilon_p) \to B(0, r)$ defined by sending $x \mapsto \frac{ra(x)}{\epsilon} - a(p)$. This is clearly a smooth map as the composition of smooth maps. Similarly, the overlap maps are smooth, for $\phi_p \circ \phi_q^{-1}(x) = \frac{r}{\epsilon_p} a \circ b^{-1}\Big(\frac{\epsilon_q}{r}(x + b(q)\Big) - a(p)$, where a, b are charts of the original atlas. Since the transition maps of the original atlas are smooth, we see that the new transition maps are smooth as well.

Finally, it is clear the the above construction yields a map corresponding to every point on M, so we see that there exists some chart sending p to the origin in some ball of radius r in $\mathbb{R}^n$.

2. Exercise 1.42

Let $[x^1,\ x^2,\ \dots,\ x^n]$ denote homogeneous coordinates. Suppose now that $x = (x^1,\ x^2,\ \dots,\ x^n) \in \phi_i(U_i) \cap \phi_j(U_j)$. Then, $\phi_j^{-1}(x) = [x^1,\ x^2,\ \dots, x^{j-1},\ 1,\ x^j, \dots,\ x^n]$, and $\phi_i([x^1,\ x^2,\ \dots, x^{J-1},\ 1,\ x^J, \dots,\ x^n]) =$

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$\left(\frac{x^1}{x^i}, \dots, \frac{x^{j-1}}{x^i}, \frac{1}{x^i}, \frac{x^j}{x^i}, \dots, \frac{x^n}{x^i}\right)$, where the ith coordinate of

$$[x^1, \ x^2, \ \dots, x^{j-1}, \ 1, \ x^j, \dots, \ x^n]$$

has been omitted in this new n-tuple.

It is obvious that this is a smooth map, since by definition of U_i, $x^i \neq 0$, and we are done.

3. Exercise 1.46

The overlap maps for the complex projective space are computed identically to those computed as above:

$$\phi_i \circ \phi_j^{-1}(z) = \left(\frac{z^1}{z^i}, \dots, \frac{z^{j-1}}{z^i}, \frac{1}{z^i}, \frac{z^j}{z^i}, \dots, \frac{z^n}{z^i}\right)$$

Now, for $\mathbb{C}P^1$, by the above, $\phi_1 \circ \phi_2^{-1}(z) = \frac{1}{z}$. From here it is clear that the domain must be $\mathbb{C}\backslash\{0\}$. Similarly, the same holds for $\phi_2 \circ \phi_1^{-1}(z) = \frac{1}{z}$, and hence as in the book's notation, $U_1 \cap U_2 = \mathbb{C}\backslash\{0\}$.

Given $\overline{\phi_1}$ as defined, noting that conjugation is idempotent, we see that $\overline{\phi_1}^{-1}(z) = [1, \overline{z}]$. Then, $\phi_2([1, \overline{z}]) = \frac{1}{\overline{z}}$, as expected.

4. Exercise 1.47

To show that we have a diffeomorphism, we must show smoothness of f and f^{-1} with respect to the maps ϕ_1, ϕ_2 defined as above. To show bijectivity, we will compute f^{-1}. Without loss of generality, we can assume all elements in $\mathbb{C}P^1$ are of the form $[z, 1]$ with the exception of $[1, 0]$. However, it is obvious that $f^{-1}([1, 0]) = (0, 0, -1) \in S^2$.

Now, for $[z, 1]$, we can find that $f^{-1}([z, 1]) = \left(\frac{2x}{1+|z|^2}, \frac{2y}{1+|z|^2}, \frac{1-|z|^2}{1+|z|^2}\right)$ if $z = x + iy$. First, let us check that this is even an element of S^2:

$$\left(\frac{2x}{1+|z|^2}\right)^2+\left(\frac{2y}{1+|z|^2}\right)^2+\left(\frac{1-|z|^2}{1+|z|^2}\right)^2 = \frac{1}{(1+|z|^2)^2}\Big(4|z|^2+(1-|z|^2)^2\Big) \tag{4.1}$$
$$= \frac{(1+|z|^2)^2}{(1+|z|^2)^2} = 1$$

So this lies on S^2. Now, we must show that this is actually f^{-1}:

$$\begin{aligned} f\circ f^{-1}([z,1]) &= f\Big(\frac{2x}{1+|z|^2}, \frac{2y}{1+|z|^2}, \frac{1-|z|^2}{1+|z|^2}\Big) \\ &= \Big[\frac{\frac{2x}{1+|z|^2}}{1+\frac{1-|z|^2}{|z|^2+1}} + \frac{\frac{2y}{1+|z|^2}i}{1+\frac{1-|z|^2}{|z|^2+1}},\ 1\Big] \\ &= \Big[\frac{2x}{1+|z|^2+1-|z|^2} + \frac{2yi}{1+|z|^2+1-|z|^2},\ 1\Big] \\ &= [x+yi,\ 1] = [z,1] \end{aligned} \tag{4.2}$$

And similarly,

$$\begin{aligned} f^{-1}\circ f(x_1,x_2,x_3) &= f^{-1}\Big[\frac{x_1}{1+x_3}+\frac{x_2 i}{1+x_3},\ 1\Big] \\ &= (x_1,x_2,x_3) \end{aligned} \tag{4.3}$$

Now, viewing elements in $\mathbb{C}$ as ordered pairs, we see that, using the charts given in the previous problem, $\phi_i\circ f:\mathbb{R}^3\to\mathbb{R}^2$, $i=1,\ 2$. Explicity, this map is $(x,y,z)\mapsto\Big(\frac{x}{1+z},\frac{y}{1+z}\Big)$ when $i=1$, assuming $z\neq -1$ (if $z=-1$, this merely gets sent to $(1,0)$).

This is clearly a smooth map, as all partial derivatives exist and are continuous since the only singularity is at $z=-1$. Similarly, for $i=2$ and $z\neq 1$ this becomes $(x,y,z)\mapsto\Big(\frac{x}{1-z},\frac{-y}{1-z}\Big)$, which is again seen to be smooth.

For the inverse, this becomes $(x,y)\mapsto\Big(\frac{2x}{1+x^2+y^2},\frac{2y}{1+x^2+y^2},\frac{1-x^2+y^2}{1+x^2+y^2}\Big)$ when $i=1$. This is again easily seen to be smooth, since all partial derivatives exist and are continuous. When $i = 2$, we see that $(x, y) \mapsto \left(\frac{2x}{(x^2+y^2)(1+y^2+x^2)}, \frac{-2y}{(x^2+y^2)(1+x^2+y^2)} \right)$, which is smooth.

Thus this is a diffeomorphism since f and f^{-1} are smooth with respect to the atlas on $\mathbb{C}P^1$, so we are done.

5. Exercise 1.48

Define $f : S^1 \to \mathbb{R}P^1$ by $[z(x, y), 1]$ for $y \neq -1$ and $[1, w(x, y)]$ for $y \neq 1$, where $z(x, y) := \frac{x}{1+y}$, $w(x, y) := \frac{x}{1-y}$. It is easily seen that when $y \neq \pm 1$, $[v(x, y), 1] = [1, w(x, y)]$.

Then, by work similar to the previous exercise, we can find $f^{-1}([t, 1]) = \left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2} \right)$, and $f^{-1}([1, 0]) = (0, -1)$. Using this and the given charts ϕ_i, $i = 1, 2$, we can find that for $i = 1$, $\phi_1 \circ f$ takes $(x, y) \mapsto \frac{x}{1+y}$ when $y \neq -1$. This is smooth since the only singularity is at $y = -1$. Similarly, for $i = 2$, $\phi_2 \circ f$ takes $(x, y) \mapsto \frac{x}{1-y}$ when $y \neq 1$. Since the only singularity is at $y = 1$, this is smooth.

For this inverse maps, if $i = 1$ this takes $t \mapsto \left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2} \right)$, which is smooth since $1 + t^2 > 0$ for all real t. Similarly, for $i = 2$, $f^{-1} \circ \phi_2$ takes $t \mapsto \frac{2t}{1+t^2}$, which is again smooth.

Thus, we have constructed a diffeomorphism between S^1 and $\mathbb{R}P^1$ and we conclude that these two manifolds are diffeomorphic.